A Stable Finite Difference method for Heat Equations

M. Shamna

Assistant Professor, Department of Mathematics
Government Women's Polytechnic College
Kayamkulam, Alappuzha

*Dr.K.Selvakumar**

Assistant Professor, Department of Mathematics
Anna University, University College of
Engineering, Nagercoil

Abstract : A stable finite difference approach to the one-dimensional heat equations is presented in this article. The suggested approach fits exponentially fitted and explicit in nature. The method's order of convergence is one. The numerical solution to Burger's equation can be found using this method. The performance of the suggested numerical method is demonstrated by the provided numerical and graphical results. This approach will be used to solve more generic non-linear equations in the future.

Key words : Heat equation, Stable method, Uniform convergence, Burgers equation.

AMS Mathematics Subject Classification : 34A07, 35R13.

1 Introduction

An equation of the form, for a given field u(x,t) and diffusion coefficient ε, in one spatial dimension was presented by Harry Baresman in 1915 [3],

$$(\frac{\partial}{\partial t}+u(x,t)\frac{\partial}{\partial x})u(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}u(x,t),\, for\, all\,(x,t)\, in\,\overline{Q}. \tag{1.1}$$

Once more, Burger employed the viscous Bateman-Burger equation (1.1) as a mathematical model to illustrate the theory of turbulence in 1948 [5]. The Burgers equation (1.1) becomes an inviscid Burgers equation of the following form when the diffusion term is removed:

$$(\frac{\partial}{\partial t}+u(x,t)\frac{\partial}{\partial x})u(x,t)=0,\, for\, all\,(x,t)\, in\, Q. \tag{1.2}$$

Using initial and boundary conditions of the type [2, 6, 7, 8, 12, 13, 14, 15, 16, 19, 20, 21, 22] for the numerical solution of a heat equation in one space variable and a time variable, we present an exponentially fitted numerical technique in this article to solve the equation (1.1).

For the numerical solution of the heat equation with non-local conditions, finite difference techniques are accessible at [1]. In [3], numerical approaches are reviewed. [4] discusses the stability of the numerical solution of the heat equation. [9] provides computational techniques for heat equations. Burger's equation can be solved numerically in [10]. [11] provides estimates of stability, convergence, and error for the numerical solution of the heat equation.

$$(\frac{\partial}{\partial t}+L_\varepsilon)u(x,t)=d(x,t),\, for\, all\,(x,t)\, in\,\overline{Q} \tag{1.3}$$

$$u(x,0)=f(x),\, on\, B_I=\{(x,0):0\le x\le 1\} \tag{1.4}$$

$$u(0,t)=g(t),\, on\, B_L=\{(0,t):0\le t\le T\} \tag{1.5}$$

$$u(1,t)=h(t),\, on\, B_R=\{(1,t):0\le t\le T\} \tag{1.6}$$

where

$$L_\varepsilon = -\varepsilon \frac{\partial^2}{\partial x^2}, \tag{1.7}$$

and $0 < \varepsilon << 1$ is a singular perturbation parameter.

The following notations will be used to analyze the singularly perturbed parabolic equation (1.3)-(1.7) solution's stability, boundedness, and uniqueness: $Q_x = (0,1)$, $Q_t = (0,T)$, $Q = Q_x \times Q_t$, $\overline{Q_x} = [0,1]$, $\overline{Q_t} = [0,T]$, $\overline{Q} = \overline{Q_x} \times \overline{Q_t}$, $B_I = (x,0) : 0 \leq x \leq 1$, $B_L = (0,t) : 0 \leq t \leq T$, $B_R = (1,t) : 0 \leq t \leq T$, $and\, B = B_I \cup B_L \cup B_R$. In $\overline{Q}$, the function d(x, t) is sufficiently smooth. In B, the functions $f(x)$, $g(t)$, $and\, h(t)$ are enough smooth. The parabolic equation (1.3)-(1.7) shows a boundary layer at $x = 1$ under the circumstances mentioned above.

These are the requirements for compatibility: $u(0,0) = f(0) = g(0)\, and\, u(1,0) = f(1) = h(0)$. The solution to equation (1.3)-(1.7) is distinct. $u(x,t) \in C^{2+\overline{\alpha}}, \alpha \in (0,1)$ We require the high regularity requirement for the solution of (1.3)-(1.7) to find error estimates for the numerical technique. At the corner points, we take $u(x,t) \in C^{4+\overline{\alpha},2+\frac{\alpha}{2}}$, $\alpha \in (0,1)$ as the strong regularity requirement.

1.1 Maximum Principle

The maximum principle is admitted by the solution to (1.3)-(1.7), which is expressed as follows:

Lemma 1. *Assume that $v(x,t) \in C^{2,1}(\overline{Q})$ such that $v(x,t) \geq 0$ for all points (x,t) in D and $(\frac{\partial}{\partial t} + L_\varepsilon)v(x,t) \geq 0$ for all points (x,t) in D. Then, $v(x,t) \geq 0$ for all points (x,t) in $\overline{Q}$*

The parabolic differential equation (1.3)-(1.7) has a bounded solution, which is expressed as follows:

Lemma 2. *The following bound is satisfied by the solution $u(x,t)$ of equation (1.3)-(1.7).*

$$\mid u(x,t) \mid \leq C \text{ for all points } (x,t) \text{ in } \overline{Q}$$

1.2 Stability Result

Lemma 3. *The estimate is satisfied by the solution $u(x,t)$ of equations (1.3)-(1.7).*

$$\parallel u(x,t) \parallel \leq \max[|f(x)|, |g(t)|, |h(t)|] + \parallel (\frac{\partial}{\partial t} + L_\varepsilon)u(x,t) \parallel$$

for all points (x,t) in $\overline{Q}$.

In [2, 6, 19, 20, 21, 22], finite difference techniques for (1.3)-(1.7) have been covered. [2, 13] provides a thorough overview of finite difference techniques for (1.3)-(1.7). The purpose of this paper is to provide a consistently convergent explicit technique for (1.3)-(1.7), and to Burgers equation.

1.3 Motivation

For (1.3)-(1.7), classical Euler's finite difference technique can be applied and it will be of the form

$$(u_{i,j+1} - u_{i,j})/k = [\varepsilon(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/h^2],\, for\, i = 1(1)N-1,\, j = 0(1)M-1, \tag{1.8}$$

On simplifying equation (1.8) we get,

$$u_{i,j+1} = [1-2\lambda]u_{i,j} + \lambda u_{i-1,j} + \lambda u_{i+1,j},\, for\, i = 1(1)N-1,\, j = 0(1)M-1, \tag{1.9}$$

where $\lambda = \frac{\varepsilon k}{h^2}$, and h and k are the step sizes taken concerning space and time variables respectively. The numerical solution of equation (1.8) s stable if and only if $\lambda \leq \frac{1}{2}$. This motivates us to present an exponential-fitted finite difference method based on the studies related to uniformly convergent exponential-fitted finite difference methods [13, 14, 15, 16] to remove this restriction on λ.

1.4 Construction of this Article

An explicit uniformly convergent exponentially fitted finite difference technique is described in section 2 for the boundary value problem (1.3)-(1.7). Numerical and graphical results are provided in section 3 using a test problem, heat equation. In section 4, Burgers equation is transformed into a linear parabolic equation and can be solved numerically using the method of section 2. The conclusion is given in section 5.

The mesh size along the x- and t-axes, denoted by h and k, is $\lambda = \varepsilon k/h^2$ throughout this paper. Furthermore, C is a constant that is unaffected by x, t, ε, h, and k.

2 An Explicit Fitted Finite Difference Method

For the parabolic differential equation (1.3)-(1.7), the finite difference method is provided in the form

$$(u_{i,j+1}-u_{i,j})/k = \sigma_{i,j}[\varepsilon(u_{i+1,j}-2u_{i,j}+u_{i-1,j})/h^2] = d(x_i,t_j),\ for\, i = 1(1)N-1,\ j = 0(1)M-1 \quad (2.1)$$

$$u_{i,0} = f(x_i)\, for\, i = 0(1)N \quad (2.2)$$

$$u_{0,j} = g(t_i)\, for\, j = 0(1)M-1 \quad (2.3)$$

$$u_{m,j} = h(t_i),\ for\, j = 0(1)M-1 \quad (2.4)$$

where $\sigma_{i,j}$, the fitting factor, is found for $i = 1(1)N-1$, $j = 0(1)M-1$, and it will be adjusted subsequently. Here $\sigma_{i,j}$ is define $\sigma_{i,j} = \frac{\sigma(-\gamma)\times\sigma(\gamma)}{\sigma(-\beta)}$ where $\sigma(-\gamma) = \gamma/[1-exp(-\gamma)]$, $\sigma(\gamma) = \gamma/[exp(\gamma)-1]$, $\sigma(-\beta) = \beta/[1-exp(-\beta)]$, $\gamma = h\pi/\sqrt{(\varepsilon)}$, and $\beta = k\pi^2$. The numerical method (2.1)-(2.4) is consistent with the parabolic differential equation (1.3)-(1.7) as the space and time steps h and k approaches zero respectively. The numerical solution is stable on applying the numerical method (2.1)-(2.4) to the problem (1.3)-(1.7). With reference to [17], the numerical method (2.1)-(2.4) is L-Stable and A-Stable. And, with reference to [18], the numerical method (2.1)-(2.4) is exponentially fitted and explicit in nature. The discrete maximum principle is admitted by the method (2.1)-(2.4) , which is expressed as follows:

2.1 Discrete Maximum Principle

Lemma 4. *Assume that $v_{i,j} \in C^{2,1}(\overline{Q})$ such that $v_{i,j} \geq 0$ for all points (x_i,t_j) in $\overline{Q}$ and $(\frac{\partial}{\partial t}+L_\varepsilon)v_{i,j} \geq 0$ for all points (x_i,t_j) in $\overline{Q}$. Then, $v_{i,j} \geq 0$ for all points (x_i,t_j) in $\overline{Q}$.*

The method (2.1)-(2.4) has a bounded solution, which is expressed as follows:

Lemma 5. *The solution $u_{i,j}$ of the method (2.1)-(2.4) satisfy the bound of the form*

$$\mid u_{i,j} \mid \leq C \text{ for all points } (x_i,t_j) \text{ in } \overline{Q}.$$

Lemma 6. *The estimate is satisfied by the solution $u_{i,j}$ of the method.*

$$\| u_{i,j} \| \leq \max[|f(x_i)|,|g(t_j)|,|h(t_j)|] + \| (\frac{\partial}{\partial t}+L_\varepsilon)u_{i,j} \|$$

for all points (x_i,t_j) in $\overline{Q}$.

The method (2.1)-(2.4) is stable unconditionally. The numerical method (2.1)-(2.4) is both consistent and stable leads to the fact that the numerical method (2.1)-(2.4) is convergent.

2.2 Uniform Convergence Result

As a result, as h and k trend to zero, the solution of the finite difference scheme (2.1)-(2.4) converges uniformly to the solution of the original boundary value issue. Therefore, the numerical method (2.1)-(2.4) is uniformly convergent at $O(k+h)$. It is possible to put this result, in this way:

Theorem 2.1. *Let* $u_{i,j}$ *and* u *be the solution of the difference method (2.1)-(2.4) and the initial boundary value problem (1.3)-(1.7) respectively. Then , for all* $0 \le x \le 1$ *and* $0 \le t \le 1$,

$$\max |u_{i,j} - u(x_i,t_i)| \le C(k+h) \tag{10}$$

where C *is independent of* i, j, k, h, *and* ε.

3 Numerical Results

This part involves applying the strategy (2.1)-(2.4) to a test problem.
Test Problem 1. Consider the heat equation [2, 6, 13, 19, 20, 21]

$$u_t(x,t) = \varepsilon u_{xx}(x,t), 0 < x < 1, t > 0 \tag{3.1}$$

$$u(x,0) = \sin(\pi x/\sqrt{(\varepsilon)}), 0 \le x \le 1, \tag{3.2}$$

$$u(0,t) = 0, u(1,t) = sin(\pi/\sqrt{(\varepsilon)}).exp(-t), t \ge 0 \tag{3.3}$$

To the test problem 1, that is, problem (3.1) - (3.3) numerical results are given in Table 1, and Table 2. Graphical results are shown in Figure 1, and Figure 2. Table 1, is only for $t = 1/8$ with $h = k = 1/8$, and ε = 1. And, the Table 2 also is only for $t = 1/8$ with $h = k = 1/8$, and ε = 1.

TABLE I: Numerical solution using classical Euler's method

Nodals	Numerical solution	Exact solution	Absolute error	Relative error
U[1,1]	1.72541E-01	1.11442E-01	6.10986E-02	5.48253E-01
U[2,1]	3.18814E-01	2.05919E-01	1.12896E-01	5.48253E-01
U[3,1]	4.16551E-01	2.69046E-01	1.47505E-01	5.48253E-01
U[4,1]	4.50871E-01	2.91213E-01	1.59658E-01	5.48253E-01
U[5,1]	4.16551E-01	2.69046E-01	1.47505E-01	5.48253E-01
U[6,1]	3.18814E-01	2.05919E-01	1.12895E-01	5.48253E-01
U[7,1]	1.72541E-01	1.11442E-01	6.10986E-02	5.48253E-01

TABLE II: Numerical solution using proposed exponentially fitted method

Nodals	Numerical solution	Exact solution	Absolute error	Relative error
U[1,1]	1.18325E-01	1.11442E-01	6.88266E-03	6.17598E-02
U[2,1]	2.18636E-01	2.05919E-01	1.27175E-02	6.17598E-02
U[3,1]	2.85662E-01	2.69046E-01	1.66162E-02	6.17598E-02
U[4,1]	3.09198E-01	2.91213E-01	1.79852E-02	6.17598E-02
U[5,1]	2.85662E-01	2.69046E-01	1.66162E-02	6.17598E-02
U[6,1]	2.18636E-01	2.05919E-01	1.27175E-02	6.17598E-02
U[7,1]	1.18325E-01	1.11442E-01	6.88266E-03	6.17598E-02

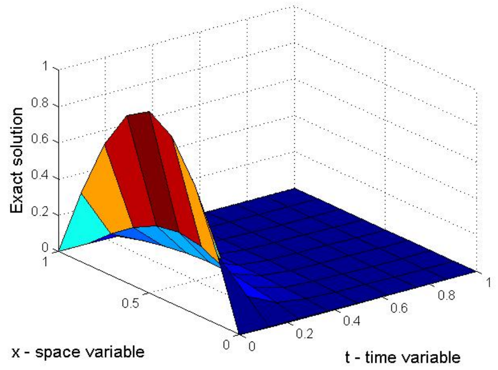

Fig. 1: Exact solution of the heat equation

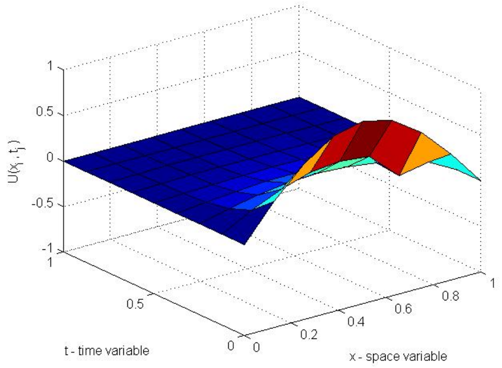

Fig. 2: Solution got by applying classical Euler's method

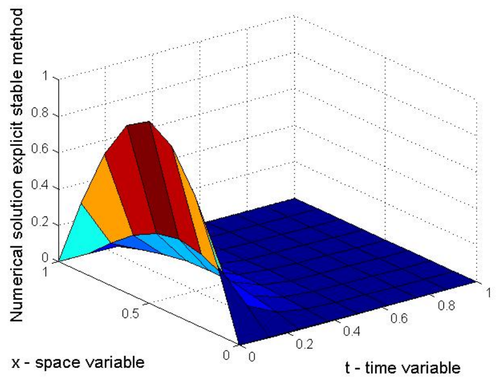

Fig. 3: Solution got by applying proposed exponentially fitted method

3.1 Observations

From Table 1, it is observed that both absolute and relative errors are not of order one. From Table 2, both absolute and relative errors are of order one convergence. Exact solution of the heat equation is given in Figure 1. From Figure 2, it is observed that as time t increases the numerical solution is not suitable. From Figure 3, it is observed that as time t increases the numerical solution approximates well to the exact solution. Numerical solution approximate the exact solution with order one.

4 Bateman-Burger Equation

We consider the Burger's equation

$$u_t + uu_x = \varepsilon u_{xx} \tag{4.1}$$

subject to the conditions (1.4)-(1.6). To solve Burger's equation (4.1), it is enough to use the simple transportation

$$u = -2\varepsilon \frac{v_x}{v} \tag{4.2}$$

The equation (4.1) will reduces to the heat equation

$$\frac{\partial}{\partial t} v(x,t) = \varepsilon \frac{\partial^2}{\partial x^2} v(x,t) \tag{4.3}$$

The heat equation (4.3) can be solved numerically using the method given in the article.

5 Conclusions

We have solved a singularly perturbed one-dimensional heat equation subject to initial and boundary conditions with a boundary layer at the right end boundary condition using an explicit method. And, using a transformation a Burgers equation is transformed into a linear equation. And, it is solved by the numerical method proposed for the singularly perturbed linear parabolic differential equation. This method can be applied to the generalized linear heat equation. This method can be applied to non-linear heat equations This method can be applied to heat equations with two space variables.

Acknowledgement

All computations were performed in Pascal single precision on a Micro Vax II computer at Bharathidasan University, Tiruchirapalli-620 024, Tamil Nadu, India. And, graphical results using Matlab at the University of Kerala, Thiruvananthapuram, and Anna University, Chennai. The authors carried out this research work.

References

[1] W.T, Ang, W. T, A solution method for the one-dimensional heat equation subject to nonlocal conditions. Southeast Asian Bulletin of Mathematics, 26, 185-191, 2003.

[2] W. F. Ames, Numerical Methods for Partial Differential Equation, Academic Press, 1977.

[3] H. Bateman, Some recent researches on the motion of fluids, Monthly Weather Review, 43(4): 163-170, 1915.

[4] N. Borovykh, Stability in the numerical solution of the heat equation with nonlocal boundary conditions. Applied numerical mathematics, 42(1-3), 17-27, 2002.

[5] J.M. Burger, A mathematical model illustrating the theory of turbulence, Advances in Applied Mechanics,1:179-199, 1948.

[6] J. L. Davis, Finite Difference Methods in Dynamics of Continuous Media, Macmillan Publishing Company, 1980.

[7] E. P. Doolan, J. J. H. Miller, and W.H. A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers. Boole Press, Dublin, 1980.

[8] Z. I. Hassen and G. F. Duressa, New approach of convergent numerical method for singularly perturbed delay parabolic convection-diffusion problems, Research in Mathematics, 10:1-14, 2023.

[9] S.R. Jena S.R, and G.S Gebremedhin, Computational technique for heat and advection diffusion equations. Soft Computing. 25(16),11139-11150, 2021.

[10] S.R. Jena, and G.S Gebremedhin, Decatic B-spline collocation scheme for approximate solution of Burgers equation. Numerical Methods for Partial Differential Equations, 9(3), 18511869, 2023.

[11] S.R. Jena, A. Senapati, Stability, convergence and error analysis of B-spline collocation with the Crank-Nicolson method and finite element methods for numerical solution Schrodinger equation arises in quantum mechanics, Physica Scripta, 98(11), 115232, 2023.

[12] M. J, OReilly, On Uniformly Convergent Finite Difference Methods for Non-linear Singular Perturbation Problems. Ph.D. Thesis, Trinity College, Dublin,1983.

[13] K. Selvakumar, Uniformly Convergent Difference Schemes for Differential Equations with a Parameter. Ph.D. Thesis, Bharathidasan University, India, 1992.

[14] K . Selvakumar, Optimal Uniform Finite Difference Schemes oh Order Two for Stiff Initial Value Problems, Int. J.Com. Numerical Methods in Engineering 10: 611-622,1994.

[15] K. Selvakumar, A Computational Methods for solving Singular Perturbation Problems using Exponentially Fitted Finite Difference Schemes. J. Appl. Math.& Comp. 80:1-14, 1994.

[16] K . Selvakumar, A Computational Method for a Chemical Flow Reactor Problem using Shooting Method., J. Appl. Math. & Comp., 68:27-40, 1995.

[17] K. Selvakumar, and K. Jason, L-stable and A-stable numerical method of order two for a stiff differential equation, Soft Computing, 26(22), 12779 - 12794, 2022.

[18] K. Selvakumar, Two new optimal and uniform third-order schemes for singular perturbation problems with initial layers, Soft Computing, Volume 26(22), 12811 - 12832, 2022.

[19] R.A. Vilkov, Numerical Methods, Mir Publishers, Moscow, 1980.

[20] N. N Yanenko and Yu.I.Shokiu, Numerical Methods in Dynamics, Mir Publishers, Moscow, 1984

[21] E. Zauderer, Partial Differential Equations of Applied Mathematics, Addison Wiley Inter Science Publication, 1983.

[22] H. Zhen, Pu.Hongyun and S. Hongjun, Pure Alternating Segment Explicit- Implicit Method for the Diffusion Equations, Int. J. Comp. Math. 55-58, 1994